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FREE NON-LINEAR VIBRATION OF A ROTATING THIN RING WITH THE IN-PLANE AND OUT-OF-PLANE MOTIONS

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Free non-linear vibration of a rotating thin ring with a constant speed is analyzed when the ring has both the in-plane and out-of-plane motions. The geometric non-linearity of displacements is considered by adopting the Lagrange strain theory for the circumferential strain instead of the infinitesimal strain theory. By using Hamilton's principle, the coupled non-linear partial differential equations are derived, which describe the out-of-plane bending and torsional motions as well as the in-plane bending and extensional motions. During deriving the equations of motion, we discuss how to model the circumferential stress and strain in order to consider the geometric non-linearity. Four models are established: three non-linear models and one linear model. For the four models, the linearized equations of motion are obtained in the neighbourhood of the steady state equilibrium position. Based on the linearized equations of the four cases, the natural frequencies are computed at various rotational speeds and then they are compared. Through the comparison, this study recommends which model is appropriate to describe the non-linear behaviour more precisely.

1. INTRODUCTION

The vibration analysis of a stationary or rotating ring is of great interest because the ring has very simple geometry but shows most of the dynamic characteristics of more complex axisymmetric structures. An axisymmetric structure, which may be modelled as a ring, is used in many engineering applications such as ring stiffeners, gears and rate-sensors.

The free and forced vibrations of stationary or rotating rings have been widely studied for various ring models and for various boundary conditions. Rao and Sundararajan [1] and Kirkhope [2] investigated the in-plane vibrations of stationary rings using linear formulations. The corresponding non-linear case was treated by Evensen [3]. In addition to the in-plane vibration analyses of stationary rings, many papers are also available for the in-plane vibrations of rotating rings. For example, the linear vibration of a rotating ring was investigated by Carrier [4] and the non-linear formulations were used by Huang and Soedel [5], Natsiavas [6], and Bickford and Reddy [7]. On the other hand, studies

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involving the out-of-plane motion can be found in the literature but most of the studies are concerned with stationary rings. For instance, Kirkhope [8] and Lee and Chao [9] presented results for stationary rings using linear models while Maganty and Bickford [10] studied the stationary ring using a non-linear model. Regarding the out-of-plane vibrations of rotating rings, one available work is that of Eley *et al.* [11] on a multi-axis rate sensor in which the Coriolis coupling effects were studied. However, they did not consider the non-linear vibration of a ring. To the authors' knowledge, the out-of-plane non-linear vibration of a rotating ring has not yet been treated. The non-linearity, caused by the coupling between the in-plane and out-of-plane displacements, should be considered in the out-of-plane vibration of a rotating ring. The reason is that this non-linearity is related to the stiffening effect of a ring due to rotation.

In this paper, free vibration of a ring rotating at a constant speed is studied, considering the non-linearity and coupling of the in-plane and out-of-plane displacements. In order to include the geometric non-linearity into the formulation, the Lagrange strain theory is adopted for the circumferential strain, instead of using the infinitesimal strain theory. By using Hamilton's principle, the non-linear equations of motion for a rotating ring are derived, which describe the out-of-plane bending and torsional motions as well as the in-plane bending and extensional motions. With the non-linear equations, an equilibrium position and the linearized equations of motion in the neighbourhood of the equilibrium position are obtained by the perturbation method. Natural frequencies are calculated from the linearized equations for various rotational speeds. Finally, computation results from various non-linear models are compared with those from a linear model and some differences among them are discussed.

2. EQUATIONS OF MOTION

Figure 1(a) shows an unconstrained ring rotating at a constant angular speed Ω about the Z-axis where the XYZ co-ordinate system is a space-fixed inertial frame. The r, θ and Z are the cylindrical co-ordinates where θ is measured from the stationary X-axis, and R is the radius of the undeformed centroidal line of the ring. Shown in Figure 1(b) is the cross-section of the ring where h and b are height and radial width, respectively, and the xyz co-ordinate system is a body-fixed rotating frame.

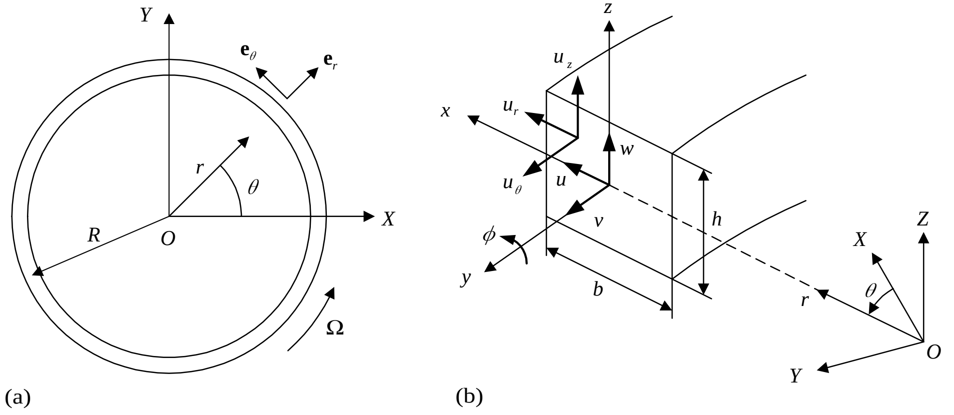

Figure 1. Unconstrained thin ring rotating at constant speed Ω: (a) top view; and (b) the cross-section.

The ring is modelled as a kind of the Euler–Bernoulli beam to study the flexural vibrations. It is assumed that the entire cross-section perpendicular to the y-axis remains a plane after deformation. Also, the ring is slender, that is, the height and radial width of the ring are much less than the radius of the ring, so that the transverse shear deformation is neglected. It is also assumed that the warping of the cross-section due to torsion is negligibly small. Therefore, the displacements of a point in the r-, θ- and z-directions can be written as

$$u_r = u(\theta, t) + z\phi(\theta, t), \; u_\theta = v(\theta, t) + x\phi_i(\theta, t) - z\phi_o(\theta, t), \; u_z = w(\theta, t) - x\phi(\theta, t), \quad (1)$$

where t is time; u, v and w are the radial, circumferential and out-of-plane displacements, respectively, of a point on the centroidal line (i.e., $x = z = 0$); ϕ is the twist angle about the y-axis due to torsion; ϕ_i is the rotation angle about the z-axis due to in-plane bending; ϕ_o is the rotation angle about the x-axis due to out-of-plane bending. According to references [1, 6, 11], the rotation angles, ϕ_i and ϕ_o, may be expressed as

$$\phi_i = (v - u')/R, \quad \phi_o = w'/R, \quad (2)$$

where the prime denotes the partial derivative with respect to θ.

The geometric non-linearity results from the large deformation of the ring. The non-linearity is generally described by the Lagrange strain theory, which represents the non-linear relations between the strains and the displacements. Based on the Lagrange strain theory in the cylindrical co-ordinates, the circumferential normal strain of a point on the centroidal line is represented by

$$\bar{\varepsilon}_\theta = \bar{\varepsilon}_\theta^L + \frac{1}{2}[(\bar{\varepsilon}_\theta^L)^2 + (\phi_i)^2 + (\phi_o)^2], \quad (3)$$

where

$$\bar{\varepsilon}_\theta^L = (v' + u)/R. \quad (4)$$

In the right-hand side of equation (3), the first term is linear and the terms in the bracket are non-linear. Note that the first non-linear term is a square of the linear term while the second and third non-linear terms are related to the in-plane and out-of-plane rotations respectively. Due to the third non-linear term, the equation of motion for the out-of-plane displacement will be coupled to those for the radial and circumferential displacements. Under the assumptions that the ring is thin and plane cross-sections remain plane after deformation, the circumferential normal strain of an arbitrary point in the ring can be approximated [5, 6, 9] by

$$\varepsilon_\theta = \bar{\varepsilon}_\theta + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi). \quad (5)$$

Note that the above relation between the circumferential normal strain and the displacements is non-linear. However, since the non-linearity of the shear deformations due to torsion is negligible, the linear relations between the shear strains and the displacements are adopted in this study. Based on reference [9], the linear shear strains can be represented by

$$\gamma_{r\theta} = \frac{z}{R}\left(\frac{w'}{R} + \phi'\right), \quad \gamma_{\theta z} = -\frac{x}{R}\left(\frac{w'}{R} + \phi'\right). \quad (6)$$

Next, consider the strain energy of the ring related to the normal and shear strains. It is assumed that the material of the ring is homogeneous, isotropic, elastic and Hookean. Young's modulus and shear modulus are given by E and G respectively. Since the height h and radial width b of the ring are very small in comparison with the radius of the ring, stresses σ_r, σ_z and τ_{zr} can be neglected. In this case, the strain energy of the ring may be expressed as

$$U = \frac{ER}{2}\int_0^{2\pi}\int_A \varepsilon_\theta^2\,\mathrm{d}A\,\mathrm{d}\theta + \frac{GR}{2}\int_0^{2\pi}\int_A(\gamma_{r\theta}^2+\gamma_{\theta z}^2)\,\mathrm{d}A\,\mathrm{d}\theta, \tag{7}$$

where A is the cross-sectional area of the ring.

To obtain the kinetic energy, the velocity of a point in the rotating ring needs to be expressed in terms of the displacements of u, v and w. The displacement vector of a point of the deformed ring can be written as

$$\mathbf{r}_P = (r+u_r)\mathbf{e}_r + u_\theta\mathbf{e}_\theta + (z+u_z)\mathbf{e}_z, \tag{8}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ represent the unit vectors of the cylindrical co-ordinate system, and note that $r = R + x$. The velocity $\mathbf{v}_P$, obtained by differentiating the displacement vector with respect to time after substituting equations (1) into (8), can be expressed as

$$\mathbf{v}_P = \frac{\mathrm{d}\mathbf{r}_P}{\mathrm{d}t} = \bar{\mathbf{v}}_P + x\mathbf{\Psi}_x - z\mathbf{\Psi}_z, \tag{9}$$

where

$$\bar{\mathbf{v}}_P = (\dot{u} + \Omega u' - \Omega v)\mathbf{e}_r + (\dot{v} + \Omega v' + \Omega u + \Omega R)\mathbf{e}_\theta + (\dot{w} + \Omega w')\mathbf{e}_z, \tag{10}$$

$$\mathbf{\Psi}_x = \frac{\Omega}{R}(u' - v)\mathbf{e}_r + \frac{1}{R}[\dot{v} - \dot{u}' + \Omega(v' - u'' + R)]\mathbf{e}_\theta - (\dot{\phi} + \Omega\phi')\mathbf{e}_z, \tag{11}$$

$$\mathbf{\Psi}_z = -\left[\frac{\Omega}{R}w' + (\dot{\phi} + \Omega\phi')\right]\mathbf{e}_r + \left[\frac{1}{R}(\dot{w}' + \Omega w'') - \Omega\phi\right]\mathbf{e}_\theta. \tag{12}$$

In the above, the superposed dot represents partial differentiation with respect to time. Then the kinetic energy of the ring may be approximated to

$$T = \frac{\rho R}{2}\int_0^{2\pi}\int_{\mathrm{A}}[\bar{\mathbf{v}}_P\cdot\bar{\mathbf{v}}_P + (x^2+z^2)(\dot{\phi} + \Omega\phi')^2 + z^2\Omega^2\phi^2]\,\mathrm{d}A\,\mathrm{d}\theta, \tag{13}$$

where ρ is the mass density of the ring. Note that the rotary inertia terms are neglected except in the torsional mode [12] because the ring is assumed to be thin.

The equations of motion can be derived by applying Hamilton's principle to equations (7) and (13). The resultant equations of motion are coupled, non-linear, partial differential equations given by

$$\ddot{u} + 2\Omega(\dot{u}' - \dot{v}) + \Omega^2(u'' - 2v' - u) - \frac{EI_i}{\rho AR^4}(v''' - u'''') + \frac{E}{\rho R}[\bar{\varepsilon}_\theta + \bar{\varepsilon}_\theta\bar{\varepsilon}_\theta^L + (\bar{\varepsilon}_\theta\phi_i)'] = R\Omega^2, \tag{14}$$

$$\ddot{v} + 2\Omega(\dot{v}' + \dot{u}) + \Omega^2(v'' + 2u' - v) - \frac{EI_i}{\rho AR^4}(v'' - u''') - \frac{E}{\rho R}[(\bar{\varepsilon}_\theta)' - \bar{\varepsilon}_\theta\phi_i + (\bar{\varepsilon}_\theta\bar{\varepsilon}_\theta^L)'] = 0, \tag{15}$$

$$\ddot{w} + 2\Omega\dot{w}' + \Omega^2 w'' + \frac{EI_o}{\rho AR^4}(w'''' - R\phi'') - \frac{GI_P}{\rho AR^4}(w'' + R\phi'') - \frac{E}{\rho R}(\bar{\varepsilon}_\theta\phi_o)' = 0, \tag{16}$$

$$\ddot{\phi} + 2\Omega\dot{\phi}' + \Omega^2\left(\phi'' - \frac{I_o}{I_P}\phi\right) - \frac{EI_o}{\rho I_P R^3}(w'' - R\phi) - \frac{G}{\rho R^3}(w'' + R\phi'') = 0, \tag{17}$$

where I_i and I_o are the area moments of inertia about the z- and x-axis, respectively, and I_P is the polar area moment of inertia about the y-axis:

$$I_i = \int_A x^2\,\mathrm{d}A, \quad I_o = \int_A z^2\,\mathrm{d}A, \quad I_P = \int_A (x^2+z^2)\,\mathrm{d}A. \tag{18}$$

Equations (14)–(17) predominantly represent the in-plane bending motion, the extensional motion, the out-of-plane bending motion and the torsional motion respectively. It is interesting that the radial and circumferential displacements, u and v, and the out-of-plane displacement, w, are coupled to each other, as shown in equations (14)–(16). Note that the last term on the left-hand side of equation (16) can be rewritten by using equations (2)–(4) as

$$-\frac{E}{\rho R}(\bar{\varepsilon}_\theta \phi_o)' = -\frac{E}{\rho R}\left\{\frac{w'(v'+u)}{R^2} + \frac{w'}{2R^3}[(v'+u)^2 + (v-u')^2 + (w')^2]\right\}'. \tag{19}$$

So, the out-of-plane displacement in equation (16) is coupled to the radial and circumferential displacements through the non-linear terms of equation (19) whereas it is coupled to the torsional displacement ϕ through the linear terms. Finally, equation (17) demonstrates that the torsional displacement is coupled to only the out-of-plane displacement.

3. NATURAL FREQUENCIES

In order to compute the natural frequencies of the rotating ring, linearized equations of motion should be obtained from equations (14)–(17) when the ring is in a steady state. The perturbation method is used to obtain an equilibrium position and the linearized equations in the neighbourhood of the equilibrium position. The displacements u, v, w and ϕ can be rewritten as

$$u = u_e + \Delta u, \quad v = v_e + \Delta v, \quad w = w_e + \Delta w, \quad \phi = \phi_e + \Delta\phi, \tag{20}$$

where u_e, v_e, w_e, and ϕ_e represent the equilibrium position and Δu, Δv, Δw and $\Delta\phi$ are the small perturbations of the displacements u, v, w and ϕ, respectively, from the equilibrium position. When the ring rotates at a constant speed Ω, the equilibrium position is defined by

$$u_e = R(\sqrt{1 + 2\rho R^2\Omega^2/E} - 1), \quad v_e = w_e = \phi_e = 0. \tag{21}$$

The linearized equations of motion in the neighbourhood of the equilibrium position may be represented in terms of Δu, Δv, Δw and $\Delta\phi$. For notational simplicity, deleting Δ from Δu, Δv, Δw and $\Delta\phi$, the linearized equations of motion are given by

$$\ddot{u} + 2\Omega(\dot{u}' - \dot{v}) + \Omega^2(u'' - 2v' - u) - \frac{EI_i}{\rho A R^4}(v''' - u'''') + \frac{E}{2\rho R^4}[(2R^2 + 6Ru_e + 3u_e^2)(v' + u) - u_e(2R + u_e)(u'' - v')] = 0, \tag{22}$$

$$\ddot{v} + 2\Omega(\dot{v}' + \dot{u}) + \Omega^2(v'' + 2u' - v) - \frac{EI_i}{\rho A R^4}(v'' - u''') - \frac{E}{2\rho R^4}[(2R^2 + 6Ru_e + 3u_e^2)(v'' + u') + u_e(2R + u_e)(u' - v)] = 0, \tag{23}$$

$$\ddot{w} + 2\Omega\dot{w}' + \Omega^2 w'' + \frac{EI_o}{\rho A R^4}(w'''' - R\phi'') - \frac{GI_P}{\rho A R^4}(w'' + R\phi'') - \frac{E}{2\rho R^4}u_e(2R + u_e)w'' = 0, \tag{24}$$

$$\ddot{\phi} + 2\Omega\dot{\phi}' + \Omega^2\left(\phi'' - \frac{I_o}{I_P}\phi\right) - \frac{EI_o}{\rho I_P R^3}(w'' - R\phi) - \frac{G}{\rho R^3}(w'' + R\phi'') = 0. \quad (25)$$

Equations (22) and (23) show that the radial displacement u and the circumferential displacement v are coupled to each other. On the other hand, from equations (24) and (25), it is observed that the out-of-plane displacement w and the torsional displacement ϕ are coupled to each other.

The natural frequencies, ω_n, can now be obtained by assuming [5, 7] that

$$u = c_1 e^{i(n\theta - \omega_n t)}, \quad v = c_2 e^{i(n\theta - \omega_n t)}, \quad w = c_3 e^{i(n\theta - \omega_n t)}, \quad \phi = c_4 e^{i(n\theta - \omega_n t)}, \quad (26)$$

which lead to characteristic equations. In the above, c_1, c_2, c_3, and c_4 are arbitrary constants, n is an integer, and $i = \sqrt{-1}$. The characteristic equation for the in-plane bending and extensional vibrations is obtained from the condition for equations (22) and (23) to have non-trivial solutions after substituting u and v of equations (26) into equations (22) and (23). The characteristic equation can then be solved numerically for the natural frequencies, provided all the parameters are specified. In general, four natural frequencies are obtained for each value of n. Two of them are predominantly associated with the in-plane bending vibration, and the other two are related to the extensional vibration. Similarly, the natural frequencies of the out-of-plane bending and torsional vibrations can be calculated from the characteristic equation obtained from the condition for equations (24) and (25) to have non-trivial solutions when w and ϕ are given by equations (26). In this case, for each value of n, two of the four natural frequencies are predominantly related to the out-of-plane bending vibration, and the other two are related to the torsional vibration.

To verify the computation of the natural frequencies, consider a numerical example of a steel ring with $\rho = 7850\,\mathrm{kg/m^3}$, $E = 207\,\mathrm{GPa}$, $G = 80\,\mathrm{GPa}$, $R = 100\,\mathrm{mm}$ and $b = h = 2\,\mathrm{mm}$. The lowest natural frequencies of each vibration mode are computed from the characteristic equations when $\Omega = 0$, and they are compared to the values obtained from Blevins [12]. The comparison of the lowest natural frequencies for the in-plane and out-of-plane bending modes and the extensional and torsional modes are shown in Table 1 where these frequencies agree well with each other. In the in-plane and out-of-plane bending modes, $n = 2$ corresponds to modes with the lowest frequencies because $n = 0$ and 1 represent the rigid-body motions of the ring. Meanwhile, in the extensional and torsional modes, $n = 0$ corresponds to the uniform radial expansion and twist modes with the lowest natural frequencies.

The natural frequencies of the rotating ring are verified and analyzed. Figure 2(a) and (b) show the natural frequencies of the in-plane and out-of-plane bending modes, respectively, as functions of Ω. In these figures, the solid and dotted lines indicate results for $n = 2$ and 3 respectively. When the ring rotates, the natural frequency of each mode for the stationary ring branches into two. In general, these branches represent the natural

TABLE 1

Lowest natural frequencies (rad/s) of each vibration mode when $\Omega = 0$

Vibration mode	In this study	Reference [12]
In-plane bending mode (when $n = 2$)	795	796
Out-of-plane bending mode (when $n = 2$)	773	756
Extension mode (when $n = 0$)	51 351	51 351
Torsion mode (when $n = 0$)	36 311	36 311

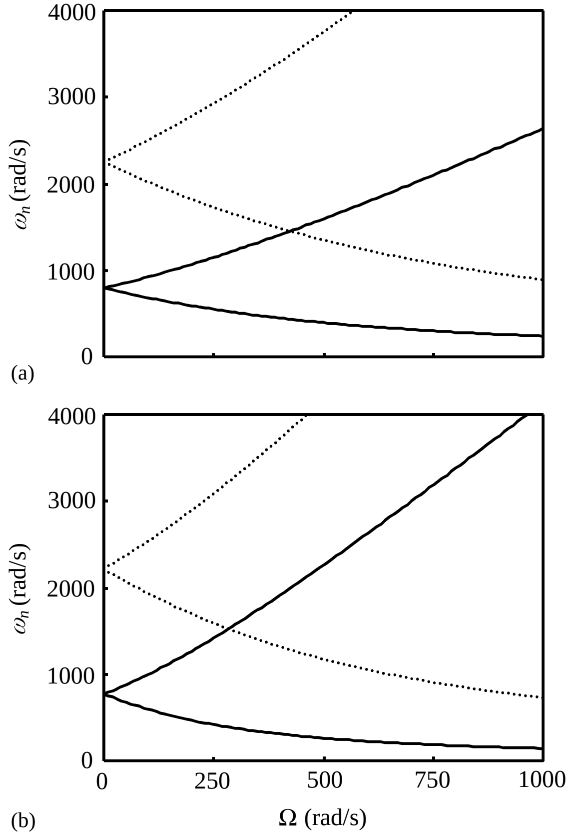

Figure 2. Natural frequencies ω_n of the bending modes versus the rotational speed Ω for (a) the in-plane vibration and (b) the out-of-plane vibration: ——, $n = 2$; ········, $n = 3$.

frequencies for the forward and backward travelling waves [5, 13]. A point to be noted in Figure 2 is that frequencies at $\Omega = 0$ for the in-plane and out-of-plane bending modes are almost the same because the cross-section of the ring is square, namely, $b = h$. However, as Ω is increased from 0, the differences between the natural frequencies for the in-plane and out-of-plane bending modes are increased. Shown in Figures 3 and 4 are the natural frequencies for the extensional and torsional modes, respectively, versus Ω. It is seen that the frequencies are much higher than those of the flexural bending vibrations. So, investigations will be focused on the flexural bending vibrations, which are of main interest from the practical point of view, in the next section.

4. EFFECTS OF THE NON-LINEAR TERMS

Some modelling issues regarding the effects of the non-linear terms in equation (3) on the flexural bending natural frequencies are discussed in this section. For clear explanation, the model of circumferential stress and strain used to obtain the non-linear equations (14)–(17) is called Model 1. As summarized in Table 2, Model 1 used the non-linear circumferential strain and stress to obtain the strain energy of the ring. Three more models shown in Table 2 will now be treated to compute the in-plane and out-of-plane flexural bending natural frequencies. Model 2, based on the von Karman strain theory,

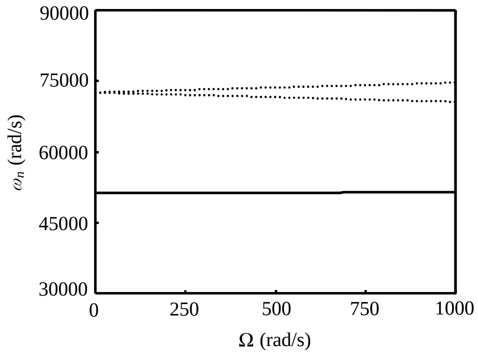

Figure 3. Natural frequencies ω_n of the extensional modes versus the rotational speed Ω: ——, $n = 0$; $\cdots\cdots\cdots$, $n = 1$.

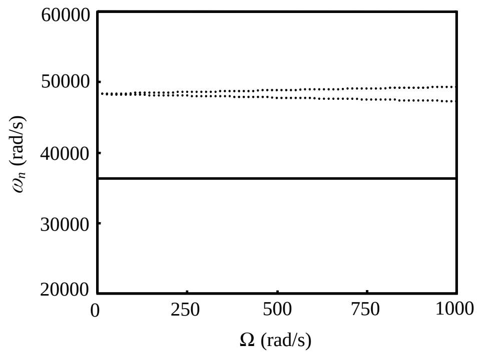

Figure 4. Natural frequencies ω_n of the torsional modes versus the rotational speed Ω: ——, $n = 0$; $\cdots\cdots\cdots$, $n = 1$.

is the case when the quadratic term of the linear normal strain, $(\bar{\varepsilon}_\theta^L)^2$, is neglected in equation (3) based on an assumption that the term is much smaller than the quadratic terms related to the in-plane and out-of-plane rotations, $(\phi_i)^2$ and $(\phi_o)^2$ respectively. The similar model of the circumferential strain for the in-plane vibration can be found in reference [6]. In Model 2, the non-linear circumferential stress is obtained by the usual manner, $\sigma_\theta = E\varepsilon_\theta$. The next model, i.e., Model 3, is the case when all non-linear strain terms in equation (3) are retained, but only the linear portion is used to get circumferential stress. As a result, Model 3 uses the non-linear circumferential strain and the linear circumferential stress; this kind of model can sometimes be found in the literature of the vibrations of rotating disks [14, 15]. The final model, Model 4, is a linear model in which the linear circumferential strain and stress are used.

The flexural bending natural frequencies are investigated for the four models shown in Table 2. Figure 5(a) and (b), for the in-plane and out-of-plane flexural bending vibrations, respectively, show the flexural bending natural frequencies ω_n when $n = 2$ as functions of the rotational speed Ω. The non-linear in-plane flexural vibration has been widely studied by many authors as mentioned in section 1. However, the results for the non-linear out-of-plane vibrations have not yet been reported in the literature to the authors' knowledge. In Figure 5, the solid, dashed and dotted lines stand for results obtained from Model 1, Model 3 and Model 4, respectively, and the circles indicate results from Model 2. From the

TABLE 2

Circumferential strain and stress of the four models used to investigate the effect of the non-linear terms on the natural frequencies

Case	Circumferential strain and stress
Model 1	$\varepsilon_\theta = \bar{\varepsilon}_\theta^L + \frac{1}{2}[(\bar{\varepsilon}_\theta^L)^2 + (\phi_i)^2 + (\phi_o)^2] + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)$
	$\sigma_\theta = E\left\{\bar{\varepsilon}_\theta^L + \frac{1}{2}[(\bar{\varepsilon}_\theta^L)^2 + (\phi_i)^2 + (\phi_o)^2] + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)\right\}$
Model 2	$\varepsilon_\theta = \bar{\varepsilon}_\theta^L + \frac{1}{2}[(\phi_i)^2 + (\phi_o)^2] + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)$
	$\sigma_\theta = E\left\{\bar{\varepsilon}_\theta^L + \frac{1}{2}[(\phi_i)^2 + (\phi_o)^2] + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)\right\}$
Model 3	$\varepsilon_\theta = \bar{\varepsilon}_\theta^L + \frac{1}{2}[(\bar{\varepsilon}_\theta^L)^2 + (\phi_i)^2 + (\phi_o)^2] + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)$
	$\sigma_\theta = E\left\{\bar{\varepsilon}_\theta^L + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)\right\}$
Model 4	$\varepsilon_\theta = \bar{\varepsilon}_\theta^L + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)$
	$\sigma_\theta = E\left\{\bar{\varepsilon}_\theta^L + \frac{x}{R}\phi_i' - \frac{z}{R}(\phi_o' - \phi)\right\}$

results presented, it can be stated in general that all four models give similar results at a very low rotational speed. For instance, the frequencies of the limiting case of a stationary ring (i.e., $\Omega = 0$) in Figure 5 can be represented by

$$\omega_n = \sqrt{\alpha - \sqrt{\alpha^2 - \beta}}, \tag{27}$$

where, for the in-plane flexural vibration,

$$\alpha = \frac{5E(4I_i + AR^2)}{2\rho AR^4}, \quad \beta = \frac{36E^2 I_i}{\rho^2 AR^6} \tag{28}$$

and, for the out-of-plane flexural vibration,

$$\alpha = \frac{EI_o(16I_P + AR^2) + 4GI_P(I_P + AR^2)}{2\rho AR^4 I_P}, \quad \beta = \frac{36EGI_o}{\rho^2 AR^6}. \tag{29}$$

Note that the same results are obtained from all models when $\Omega = 0$. However, the differences between some models become significant as Ω increases. These differences are

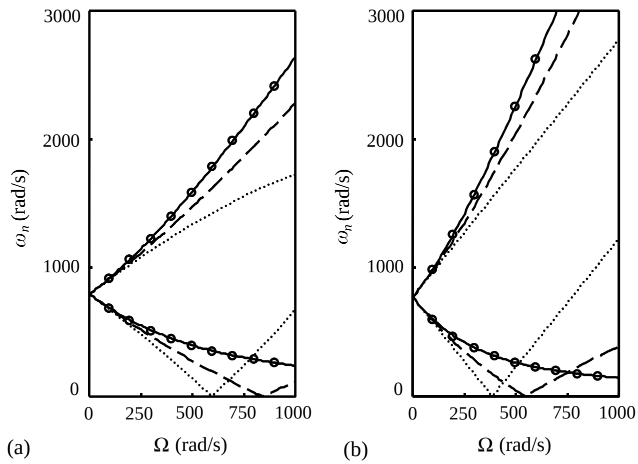

Figure 5. Comparisons of the natural frequencies ω_n of the in-plane and out-of-plane bending modes when $n = 2$: (a) the in-plane vibration; (b) the out-of-plane vibration; ———, Model 1; o o o o, Model 2; ––––, Model 3; $\cdots\cdots\cdots$, Model 4.

due to the existence of the centrifugal force. This phenomenon usually called the stiffening effect can also be observed for rotating disks, plates, and beams [16]. Note that the stiffening effect in the rotating ring could not be captured without considering the non-linear terms of equation (3). With this in mind, it can be suggested to use Model 1 or 2 instead of the simplified Model 3 or 4 for predicting the flexural bending natural frequencies at a high rotational speed. It is also seen that there is no large difference between the results obtained from Model 1 and Model 2. So, it may be beneficial to use Model 2 instead of Model 1 because the former is simpler than the latter. Finally, it is interesting to note the study of Endo *et al.* [13] for the non-linear in-plane vibrations. In their work, it was ascertained by theoretical and experimental investigations for the in-plane flexural bending modes of a rotating ring that no instability phenomenon exists, in the sense that the frequencies never become zero over the whole range of the rotational speed. Similarly, the natural frequencies for the out-of-plane bending vibrations of Models 1 and 2 in this study do not become zero over the practical operation ranges of the rotational speed, as shown in Figure 5(b).

It is now valuable to check whether or not the in-plane natural frequencies obtained from Model 2 are consistent with the results in the literature. The equation for the in-plane natural frequency found in references [5, 13] may be written as

$$\omega_n = \left| \frac{2n}{n^2+1}\Omega \pm \sqrt{\frac{n^2(n^2-1)^2}{(n^2+1)}\left(\frac{\Omega^2}{n^2+1}+K\right)} \right|, \tag{30}$$

where

$$K = \frac{Eb^2}{12\rho R^4}. \tag{31}$$

Although equation (30) was given under the assumption of inextensional deformation, the results were verified by the experiments in reference [13]. It should also be noted that equation (30) is obtained from the equations of motion derived when the ring is observed in the rotating co-ordinate system. So, the equations of motion for Model 2 have to be

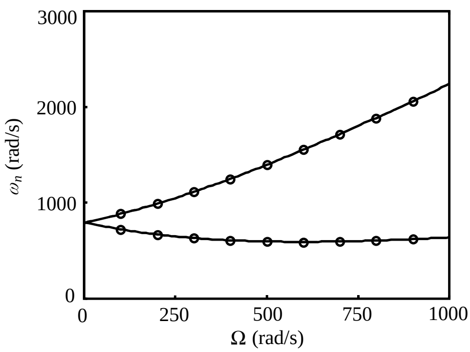

Figure 6. Comparison of the in-plane bending natural frequencies ω_n for $n = 2$ when the ring is observed in the rotating co-ordinate system: ———, reference [13]; o o o o, Model 2.

derived in the rotating co-ordinate system (i.e., θ is measured from the ring-fixed rotating axis) to compare the natural frequencies of the model with equation (30). In this case, equation (13) should be replaced by

$$T = \frac{\rho R}{2} \int_\theta \int_A [\tilde{\mathbf{v}}_P \cdot \tilde{\mathbf{v}}_P + (x^2 + z^2)\dot{\phi}^2 + z^2 \Omega^2 \phi^2] \, \mathrm{d}A \, \mathrm{d}\theta, \tag{32}$$

where

$$\tilde{\mathbf{v}}_P = (\dot{u} - \Omega v)\mathbf{e}_r + (\dot{v} + \Omega u + \Omega R)\mathbf{e}_\theta + \dot{w}\mathbf{e}_z. \tag{33}$$

In Figure 6, which shows the in-plane flexural bending frequencies versus Ω, the natural frequencies represented by the circles are obtained by the same procedure used to get the results of Model 2 in Figure 5(a). As shown in Figure 6, the frequencies of Model 2 coincide with those of equation (30) plotted by the solid line. Therefore, since the in-plane natural frequencies of Model 2 agree well with the results verified by the experiments, the model may be used to predict effectively the bending natural frequencies of the non-linear out-of-plane vibration.

5. SUMMARY AND CONCLUSIONS

Equations of motion for a ring rotating at a constant speed have been derived. To account properly for the effects of the stress generated by the centrifugal force due to rotation, the non-linear circumferential strain–displacement relation instead of the infinitesimal strain theory are used. The final governing equations derived by Hamilton's principle are coupled, non-linear, partial differential equations that cover the out-of-plane flexural bending and torsional motions as well as the in-plane flexural bending and extensional motions.

In order to compute the natural frequencies from the non-linear equations, the linearized equations of motion at the equilibrium position are obtained by the perturbation method. The natural frequencies of the rotating ring are calculated from the linearized equations at various rotational speeds. It is shown that the natural frequency of each mode for the stationary ring branches into two when the ring rotates.

Finally, several non-linear models for the circumferential stress and strain are discussed to determine which model is appropriate to describe the non-linear behaviour more precisely. Focusing on the flexural bending natural frequencies, the non-linear models are compared with the linear model. At a very low rotational speed all models give similar results, but the significant differences are observed due to the stiffening effect as a speed increases. Therefore, it is suggested to use Model 1 or Model 2 with the non-linear stress and strain for predicting the bending natural frequencies at a high rotational speed. It is also found in these models that dynamic instability does not exist over the practical ranges of a rotational speed.

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